ONE-DIMENSIONAL FLOWS OF A NONEQUILIBRIUM PLASMA WITH VARIABLE DEGREE OF IONIZATION IN THE ABSENCE OF CURRENTS

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The one-dimensional flows of an inviscid plasma not in thermal equilibrium and with a variable degree of ionization are investigated in the absence of currents. A criterion showing when the ordinary equations of gasdynamics may be used to describe these flows is given. An expression is found for the velocity of sound in such a plasma. Under certain conditions it passes into Newton's formula for isothermal sound. The condition fulfilled in the critical cross section of the channel is found. It is established that the flow of a weakly ionized plasma occurs at constant electron temperature. A detailed investigation is made of the possible types of flow in a cylindrical channel.

A criterion is given which shows when the model of a plasma in thermal equilibrium may be applied, and also relationships which permit complete calculation of the flow of such a plasma in a channel of variable cross section.

Generally speaking, the flow of a plasma with no currents present differs from the flow of a nonionized gas. This difference is related to the processes of ionization and recombination taking place in the plasma. The electrons usually play the main part in these processes, and so the average energies (temperatures) of electrons and heavy particles (atoms and ions) may differ. If the inelastic collision frequency in the plasma is small compared with the elastic collision frequency, then the temperature difference between the electron component and the heavy component of the plasma may be considerable.

The simplest cases of one-dimensional plasma flows are considered with account for ionization and recombination processes and in the absence of thermal equilibrium among the components.

1. Initial assumptions and the system of equations. We will consider the one-dimensional steady-state flow of an inviscid plasma not in thermal equilibrium using the two-fluid approximation. It is assumed that the electrons have a Maxwellian velocity distribution and that the time for the plasma to arrive at a state of thermodynamic equilibrium is much less than the characteristic times of the other processes under consideration. It is assumed for simplicity that the plasma is composed of a monatomic gas, while direct ionization by electron impact is important:

$$(n) + (e) \rightarrow (i) + 2\,(e).$$

For thermodynamic equilibrium this process should, by the principle of detailed balancing, be balanced by the reverse recombination process:

$$(i) + 2\,(e) \rightarrow (n) + (e).$$

Since ionization comes about in the forward process on account of the electron energy, the energy released in the reverse process upon recombination is also transferred to the electron. The inelastic processes in the plasma, if they are of such a nature, are determined by the electron temperature alone. Thus, from the point of view of a process changing the plasma composition, we may in this case consider a plasma which is not in thermal equilibrium as a plasma in thermal equilibrium with a temperature equal to the electron temperature in the nonequilibrium plasma, and determine its concentration by means of Saha's equation from the electron temperature (see also [1]).

The system of macroscopic equations for the plasma components is given in the general case in [2], for example. For our case the equation of conservation of the number of heavy particles, Saha's equation, and the equations of state, motion and energy for the plasma as a whole and for the electron component separately may be written in the form

$$\begin{gathered}
\frac{d}{dx}(nvS) = 0, \qquad \frac{n_e^2}{n - n_e} = 2\,\frac{g_i}{g_n}\,\frac{(2\pi m_e k t_e)^{3/2}}{h^3}\exp\left(-\frac{eU_i}{kt_e}\right), \\
P = nkT, \quad \left(P \equiv p + p_e,\ T \equiv t + \frac{n_e t_e}{n}\right), \qquad p_e = n_e k t_e, \\
mnv\,\frac{dv}{dx} + \frac{dP}{dx} = 0, \qquad \frac{dp_e}{dx} = -en_e E_x, \qquad (1.1) \\
\frac{d}{dx}\left[nvS\left(\frac{mv^2}{2} + \frac{5}{2}\,kt\right) + n_e vS\left(\frac{5}{2}\,kt_e + eU_i\right)\right] = 0, \\
\frac{d}{dx}\left[n_e vS\left(\frac{5}{2}\,kt_e + eU_i\right)\right] = -n_e vSeE_x - \delta\,\frac{m_e}{m}\,\frac{n_e S}{\tau_e}\,k\,(t_e - t).
\end{gathered}$$

Here S is the area of the channel cross section, g_i and g_n are the statistical weights of the quantum states of the ion and the neutral atom, t is the temperature of the plasma component, T is the generalized temperature, U_i is the ionization potential of the atom, k and h are Boltzmann's and Planck's constants, P is the plasma pressure, E_x is the polarization field strength in the plasma, τ_e is the relaxation time for the electron momentum, δ is a constant ~ 1.

The variables without indices relate to the heavy component of the plasma, and those with the index e to the electrons. The dielectric constant and magnetic permeability of the plasma are taken as being equal to the values in a vacuum.

We shall make a short remark concerning the polarization field in a plasma. Since we may neglect the inertia of the electrons in comparison with the heavy particle inertia, the forces on the electrons should mutually balance. The action of the electron gas pressure gradient is compensated by the polarization field which arises as a result of the separation of charges in the initial nonequilibrium period of motion.

If $dp_e/dx < 0$, then the polarization field strength $E_x > 0$. Moving in this field the electrons expend their energy, and the ions, and consequently the entire heavy component, acquire an energy $-v\,dp_e/dx$ per unit volume per second. If $dp_e/dx > 0$, then $E_x < 0$, and the heavy component loses, the electrons acquiring the energy indicated above. Energy exchange through the compensating polarization field occurs without dissipation as distinct from energy exchange through pair interactions of charged particles.

2. General analysis of the equations. Speed of sound. It is easily seen that Eqs. (1.1) pass into the ordinary equations of gasdynamics for

$$n_e / n \equiv \alpha < kt / eU_i . \qquad (2.1)$$

Here α is the degree of ionization of the plasma.

Eliminating the variables n, P, p_e and E_x from Eqs. (1.1) and solving the resulting equations with respect to the derivatives, we obtain

$$(M^2 - M^{*2}) \frac{1}{v} \frac{dv}{d\zeta} =$$

$$= -\frac{2}{5} \delta \frac{\alpha(\theta - 1)}{1 + \alpha\theta} \left(U + \frac{3}{2}\right) \frac{Z}{\xi} + \frac{M^{*2}}{S} \frac{dS}{d\zeta}, \qquad (2.2)$$

$$(M^2 - M^{*2}) \frac{1}{t} \frac{dt}{d\zeta} = \frac{2}{3} \Big[(M^2 - 1)\xi +$$

$$+ \frac{2}{5} \left(U + \frac{3}{2} \frac{1}{1 + \alpha\theta}\right) \Big] \delta\alpha(\theta - 1) \frac{Z}{\xi} - \frac{2}{3} \frac{M^2}{S} \frac{dS}{d\zeta}, \qquad (2.3)$$

$$(M^2 - M^{*2}) \frac{1}{t_e} \frac{dt_e}{d\zeta} = -(M^2 - M^{*2}) \frac{1}{U} \frac{dU}{d\zeta} =$$

$$= -\left[\frac{2 - \alpha}{1 - \alpha} (M^2 - 1) - \frac{2}{5} \frac{\alpha\theta U}{1 + \alpha\theta}\right] \delta \frac{\theta - 1}{\theta U} \frac{Z}{\xi} -$$

$$- \frac{1}{\xi} \left[1 + \frac{7 - 5\alpha}{2U(1 - \alpha)}\right] \frac{M^2}{S} \frac{dS}{d\zeta}, \qquad (2.4)$$

$$(M^2 - M^{*2}) \frac{1}{\alpha} \frac{d\alpha}{d\zeta} = -(M^2 - 1) \Big(U +$$

$$+ \frac{3}{2}\Big) \delta \frac{\theta - 1}{\theta U} \frac{Z}{\xi} - \frac{1}{\xi} \frac{M^2}{S} \frac{dS}{d\zeta},$$

$$M^2 \equiv \frac{3}{5} \frac{mv^2}{kT}, \quad \theta \equiv \frac{t_e}{t}, \quad U \equiv \frac{eU_i}{kt_e}, \quad \zeta \equiv \frac{x}{L}, \qquad (2.5)$$

$$Z \equiv \frac{m_e}{m} \frac{L}{v\tau_e}, \qquad \xi \equiv U + 3 + \frac{3}{4} \frac{7 - 5\alpha}{U(1 - \alpha)},$$

$$M^{*2} \equiv 1 - \frac{2}{5} \frac{\alpha\theta}{1 + \alpha\theta} \frac{U}{\xi}.$$

Here M^2 is a dimensionless variable analogous to the Mach number; θ is a dimensionless variable characterizing the degree to which the plasma is in thermal nonequilibrium; U is a dimensionless variable inversely proportional to the electron temperature; ζ is the dimensionless coordinate along the channel axis; L is a characteristic dimension of the channel; Z is a dimensionless variable characterizing the friction between the plasma components; M* is a critical value.

It can easily be seen that the critical value of M^2 is always situated in the interval from 3/5 to 1, i.e., between the isothermal and adiabatic values. Equating M^2 and M^{*2}, we find an expression for the critical velocity or the velocity of sound:

$$a^2 = \left(1 - \frac{2}{5} \frac{\alpha\theta}{1 + \alpha\theta} \frac{U}{\xi}\right) \frac{5}{3} \frac{P}{\rho} = \left[1 + \alpha\theta \left(1 - \frac{2}{5} \frac{U}{\xi}\right)\right] \frac{5}{3} \frac{p}{\rho}. \qquad (2.6)$$

Here $\rho \equiv mn$ is the plasma density.

If $p_e \ll p$ ($\alpha\theta \ll 1$), then the velocity of sound in the plasma coincides with the velocity of sound in an ordinary gas, as might be expected. In the second limiting case when $\alpha\theta \gg 1$ and $U \gg 1$, formula (2.6) passes over into Newton's formula for isothermal sound $a^2 = P / \rho$. This is explained by the fact that in this case the plasma pressure is determined by the pressure of the electron component. However, on the assumption, already made, that the equilibrium concentration is rapidly attained in the plasma, and for $U \gg 1$, the processes of ionization and recombination taking place in the plasma are very sensitive to the electron temperature and stabilize it. In the remaining cases the velocity of sound in the plasma lies between the isothermal and adiabatic values.

For $M^2 = M^{*2}$ the right sides of Eqs. (2.2)–(2.5) vanish simultaneously on fulfillment of the condition

$$\frac{1}{S} \frac{dS}{d\zeta} = \frac{2}{5} \frac{\delta}{M^{*2}} \frac{\alpha(\theta - 1)}{1 + \alpha\theta} \left(U + \frac{3}{2}\right) \frac{Z}{\xi}. \qquad (2.7)$$

Thus the critical cross section in the flow of a nonequilibrium plasma is not situated in the throat of the channel, but where the variables satisfy condition (2.7).

If $t_e > t$, then the critical cross section is situated in the expanding part of the channel, and if $t_e < t$ in the contracting part.

We obtain the equations for the dimensionless variables M^2 and θ:

$$(M^2 - M^{*2}) \frac{1}{M^2} \frac{d(M^2)}{d\zeta} =$$

$$= -\frac{2}{3} \left(M^2 + \frac{3}{5}\right) \left(U + \frac{3}{2}\right) \delta \frac{\alpha(\theta - 1)}{1 + \alpha\theta} \frac{Z}{\xi} +$$

$$+ \left[\frac{5}{3} M^2 \left(M^{*2} - \frac{3}{5}\right) + 2M^{*2}\right] \frac{1}{S} \frac{dS}{d\zeta}, \qquad (2.8)$$

$$(M^2 - M^{*2}) \frac{1}{\theta} \frac{d\theta}{d\zeta} = -\Big\{(M^2 - 1) \left[\frac{2}{3} d\theta\xi + \frac{2 - \alpha}{U(1 - \alpha)}\right] +$$

$$+ \frac{4}{15} \alpha\theta U\Big\} \delta \frac{\theta - 1}{\theta} \frac{Z}{\xi} + \frac{2}{3} \left(U + \frac{3}{2}\right) \frac{1}{\xi} \frac{M^2}{S} \frac{dS}{d\zeta}. \qquad (2.9)$$

The system of equations (2.4), (2.5), (2.8), and (2.9) is equivalent to the system of equations (2.2)–(2.5).

It follows from Eq. (2.9) that the plasma can remain in a state of thermal equilibrium only when flowing in a cylindrical channel.

We obtain the following expression for the polarization field in the plasma:

$$E_x = \frac{kt_e}{eL\xi} \Big\{\left[1 + \frac{7 - 5\alpha}{2U(1 - \alpha)}\right] \delta \frac{\theta - 1}{\theta} Z +$$

$$+ \frac{M^2}{M^2 - M^{*2}} \left[U + 5 + \frac{5}{4} \frac{7 - 5\alpha}{U(1 - \alpha)}\right] \times$$

$$\times \left[\frac{1}{S} \frac{dS}{d\zeta} - \frac{2}{5} \frac{\delta}{M^{*2}} \frac{\alpha(\theta - 1)}{1 + \alpha\theta} \left(U + \frac{3}{2}\right) \frac{Z}{\xi}\right]\Big\}. \qquad (2.10)$$

From the equations of conservation of the number of heavy particles, the energy of the plasma as a whole (1.1), and Saha's equation, we obtain relationships between the dimensionless variables:

$$\frac{1 + \alpha\theta}{\theta U} (M^2 + 3) + \frac{6}{5} \alpha = \text{const},$$

$$\frac{\alpha^2}{1 - \alpha} = \frac{\nu S}{U^2 \exp U} \left[\frac{1 + \alpha\theta}{\theta} M^2\right]^{1/2}, \qquad (2.11)$$

$$\left(\nu \equiv 2 \frac{g_i}{g_n} \left(\frac{5}{3m}\right)^{1/2} \frac{(2\pi m_e)^{3/2} (eU_i)^2}{h^3 n_1 v_1 S_1}\right).$$

Setting the expression for $\theta U/(1+\alpha\theta)$ from the first equation of (2.11) into (2.6), we obtain

$$a^2 = \frac{M_1^2+3+6(\alpha_1-\alpha-1/\xi)\theta_1 U_1/5(1+\alpha_1\theta_1)}{M_1^2+3+6(\alpha_1-\alpha+1/3\xi)\theta_1 U_1/5(1+\alpha_1\theta_1)}\frac{5}{3}\frac{P}{\rho}. \quad (2.12)$$

3. Flow of weakly ionized plasma. If $p_e \ll p$ $(\alpha\theta \ll 1)$, but

$$\frac{n_e}{n}\frac{eU_i}{kt} \gg 1 \ (\alpha\theta U \gg 1),$$

then Eqs. (2.2)–(2.5), (2.8), and (2.9) simplify in the following manner:

$$t_e = \text{const} \qquad (U = \text{const})$$

$$(M^2-1)\frac{1}{v}\frac{dv}{d\zeta} = -\frac{2}{5}\delta\alpha(\theta-1)Z + \frac{1}{S}\frac{dS}{d\zeta},$$

$$(M^2-1)\frac{1}{t}\frac{dt}{d\zeta} = -(M^2-1)\frac{1}{\theta}\frac{d\theta}{d\zeta} =$$

$$= \frac{2}{3}\left(M^2-\frac{3}{5}\right)\delta\alpha(\theta-1)Z - \frac{2}{3}\frac{M^2}{S}\frac{dS}{d\zeta},$$

$$\frac{1}{\alpha}\frac{d\alpha}{d\zeta} = \delta\frac{\theta-1}{\theta U_1}Z,$$

$$(M^2-1)\frac{1}{M^2}\frac{d(M^2)}{d\zeta} = -\frac{2}{3}\left(M^2+\frac{3}{5}\right)\times$$

$$\times\delta\alpha(\theta-1)Z + \frac{2}{3}(M^2+3)\frac{1}{S}\frac{dS}{d\zeta}. \quad (3.1)$$

Thus plasma flow under these conditions occurs at a constant electron temperature. An explanation of the causes of electron temperature stabilization is given above. In this case the degree of ionization α increases for $t_e > t$ and decreases for $t_e < t$.

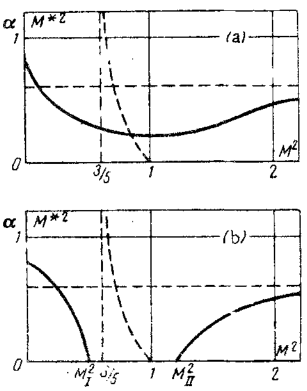

Fig. 1

Here the condition in the critical cross section has the form

$$\frac{1}{S}\frac{dS}{d\zeta} = \frac{2}{5}\delta\alpha(\theta-1)Z, \quad (3.2)$$

and the polarization field strength in the plasma is determined by

$$E_x = \frac{kt_{e1}}{eL}\left\{\delta\frac{\theta-1}{\theta U_1}Z + \right.$$

$$\left. + \frac{M^2}{M^2-1}\left[\frac{1}{S}\frac{dS}{d\zeta} - \frac{2}{5}\delta\alpha(\theta-1)Z\right]\right\}. \quad (3.3)$$

In the case under consideration relations (2.11) simplify as follows:

$$\frac{M^2+3}{\theta U_1} + \frac{6}{5}\alpha = \text{const}, \qquad \frac{\alpha^2}{S}\left(\frac{\theta}{M^2}\right)^{1/2} = \text{const} = \nu. \quad (3.4)$$

The latter relation reduces to the inverse proportionality of the degree of ionization α to the square root of its density.

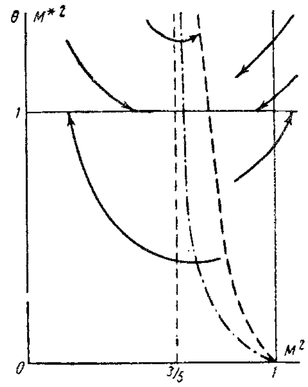

Fig. 2

4. Flow in a cylindrical channel (S = const). In this case the following integral holds:

$$mnv^2 + P = \text{const} \quad \text{for} \quad \left(\frac{1+\alpha\theta}{\theta U M^2}\right)^{1/2}\left(M^2+\frac{3}{5}\right) = \text{const}. \quad (4.1)$$

Thus we find from (2.11)

$$\alpha = \alpha_1 + \frac{5}{6}\frac{1+\alpha_1\theta_1}{\theta_1 U_1}\left[M_1^2+3-(M^2+3)\frac{M^2}{M_1^2}\left(\frac{M_1^2+3/5}{M^2+3/5}\right)^2\right],$$

$$U^{3/2}\exp U = \frac{6}{5}\nu\left(\frac{\theta_1 U_1 M_1^2}{1+\alpha_1\theta_1}\right)^{1/2}\frac{M^2}{M_1^2}\frac{M_1^2+3/5}{M^2+3/5}\times$$

$$\times\{6(1-\alpha_1)\theta_1 U_1/5(1+\alpha_1\theta_1) - M_1^2 - 3 +$$

$$+ (M^2+3)M^2(M_1^2+3/5)^2/M_1^2(M^2+3/5)^2\}\times$$

$$\times\{[M_1^2+3+6\alpha_1\theta_1 U_1/5(1+\alpha_1\theta_1) -$$

$$-(M^2+3)M^2(M_1^2+3/5)^2/M_1^2(M^2+3/5)^2]^2\}^{-1}. \quad (4.2)$$

In Fig. 1 the continuous lines represent $\alpha = \alpha(M^2)$, and the broken lines $M^{*2} = M^{*2}(M^2)$, while

$$\alpha(0) = \alpha_1 + \frac{5}{6}\frac{1+\alpha_1\theta_1}{\theta_1 U_1}(M_1^2+3),$$

$$\alpha(\infty) = \alpha_1 + \frac{3}{2}\frac{1+\alpha_1\theta_1}{\theta_1 U_1}\left(1-\frac{1}{5M_1^2}\right).$$

If

$$\alpha_1 - \frac{15}{32}\frac{1+\alpha_1\theta_1}{\theta_1 U_1 M_1^2}(M_1^2-1)^2 > 0,$$

then α has a minimum for $M^2 = 1$, while

$$\alpha_{\min} = \alpha_1 - \frac{15}{32}\frac{(1+\alpha_1\theta_1)(M_1^2-1)^2}{\theta_1 U_1 M_1^2}$$

(Fig. 1a). In the opposite case the attainable values of M^2 lie outside the interval included between the roots M_I^2 and M_{II}^2 of the equation

$$\left[5M_1^2\left(1+\frac{2}{3}\frac{\alpha_1\theta_1U_1}{1+\alpha_1\theta_1}\right)-1\right]M^4-$$

$$-\left(5M_1^4+3-4\frac{\alpha_1\theta_1U_1}{1+\alpha_1\theta_1}\right)M^2+$$

$$+M_1^2\left(M_1^2+3+\frac{6}{5}\frac{\alpha_1\theta_1U_1}{1+\alpha_1\theta_1}\right)=0. \qquad (4.3)$$

On attaining thermal equilibrium ($\theta = 1$), the plasma flow in a cylindrical channel becomes steady-state flow.

We can easily obtain a general picture of the possible plasma flows in a cylindrical channel (Fig. 2) from the differential equations (2.8) and (2.9). The continuous curves with arrows in Fig. 2 represent $\theta = \theta(M^2)$ for different types of flow, the broken curve represents $M^{*2} = M^{*2}(M^2)$, and the dash-dot curve

$$d\theta \sim M^2 - 1 + 2\alpha\theta U / 5\alpha\theta\xi + 3(2-\alpha)/2U(1-\alpha)] = 0.$$

For $t_e < t$ the plasma always passes into a state of thermal equilibrium when the channel is sufficiently long. Here subsonic flow is retarded, supersonic flow is accelerated, and the limiting values of M^2 (for $\theta =$ $= 1$) lie outside the interval

$$1-\frac{2}{5}\frac{\alpha U}{\alpha\xi+3(2-\alpha)/2U(1-\alpha)} \div 1-\frac{2}{5}\frac{\alpha}{1+\alpha}\frac{U}{\xi}=M^{*2}.$$

As is clear from Fig. 2, in subsonic flow the plasma may, for $t_e > t$, either pass into a state of thermal equilibrium or, if it does not attain a state of thermal equilibrium in a channel of any length, attain sonic velocity at the channel exit.

For $t_e > t$ in supersonic flow the plasma may attain a state of thermal equilibrium either in a continuous flow or across a discontinuity.

The steady-state flow parameters ($\theta = 1$) M^{**2}, U^{**}, and α^{**} are in all cases determined by the relationships

$$U^{**}=\left[\frac{5}{6}(M_1^2+3)+\frac{(1+\alpha_1)\theta_1U_1}{1+\alpha_1\theta_1}\right]\frac{M_1^2}{M^{**2}}\left(\frac{M^{**2}+3/5}{M_1^2+3/5}\right)^2-$$

$$-\frac{5}{6}(M^{**2}+3), \qquad U^{**3/2}\exp U^{**}=$$

$$=\nu\left(\frac{\theta_1U_1M_1^2}{1+\alpha_1\theta_1}\right)^{1/2}\frac{M^{**2}}{M_1^2}\frac{M_1^2+3/5}{M^{**2}+3/5}\times$$

$$\times\{(1-\alpha_1)\theta_1U_1/(1+\alpha_1\theta_1)-5(M_1^2+3)/6+$$

$$+5(M^{**2}+3)M^{**2}(M_1^2+3/5)^2/6M_1^2(M^{**2}+3/5)^2\}\times$$

$$\times\{[5(M_1^2+3)/6+\alpha_1\theta_1U_1/(1+\alpha_1\theta_1)-$$

$$-(M^{**2}+3)M^{**2}(M_1^2+3/5)^2/M_1^2(M^{**2}+3/5)^2]^2\}^{-1}.$$

$$\alpha^{**}=\alpha_1+\frac{5}{6}\frac{1+\alpha_1\theta_1}{\theta_1U_1}\left[M_1^2+3-\right.$$

$$\left.-(M^{**2}+3)\frac{M^{**2}}{M_1^2}\left(\frac{M_1^2+3/5}{M^{**2}+3/5}\right)^2\right]. \qquad (4.4)$$

When the velocity of sound is attained at the end of the channel, the critical plasma flow parameters U^*, α^*, and θ^* are found from the initial values of the variables using the relationships

$$\frac{3}{10\xi^*}+\frac{(\alpha^*U^*)^3\exp 2U^*}{\nu^2(1-\alpha^*)^2}=\frac{1+\alpha_1\theta_1}{\alpha^*\theta_1U_1}\frac{M_1^2+3}{4}+\frac{3}{10}\left(\frac{\alpha_1}{\alpha^*}-1\right),$$

$$\left[\frac{1+\alpha_1\theta_1}{\alpha^*\theta_1U_1}\frac{M_1^2+3}{4}+\frac{3}{10}\left(\frac{\alpha_1}{\alpha^*}-1\right)-\frac{3}{20\xi^*}\right]\frac{\nu(1-\alpha^*)}{\alpha^*U^{*3/2}\exp U^*}=$$

$$=\frac{5}{8}\left(\frac{1+\alpha_1\theta_1}{\theta_1U_1M_1^2}\right)^{1/2}\left(M_1^2+\frac{3}{5}\right),$$

$$\frac{1}{\theta^*}=\left(\frac{2}{5}\frac{U^*}{\xi^*}-1\right)\alpha^*+\frac{(\alpha^*U^*)^4\exp 2U^*}{\nu^2(1-\alpha^*)^2}. \qquad (4.5)$$

5. Plasma in thermal equilibrium ($\theta = 1$). As pointed out above, the dimensionless variable Z entering into Eqs. (2.2)–(2.5) characterizes the friction among the plasma components. If $Z \ll 1$, we may neglect this friction.

If $Z \gg 1$, then we should have $|\theta - 1| \ll 1$. if the terms on the right sides of these equations are of identical order of magnitude. This case corresponds to the model of a plasma in thermal equilibrium. The criterion of applicability for the model of a plasma in thermal equilibrium is found more accurately as follows. Assuming $d\theta/d\zeta = 0$ in equation (2.9), we find

$$\theta-1=\frac{2}{3}\frac{(U+3/2)M^2}{\delta Z\alpha\eta(M^2-M_*^2)}\frac{1}{S}\frac{dS}{d\zeta},$$

$$\eta\equiv U+3+\frac{15}{4U}+\frac{3}{U\alpha(1-\alpha)}, \qquad M_*^2=1-\frac{2}{5}\frac{U}{\eta}. \qquad (5.1)$$

Thus we obtain the necessary condition for the model to be applicable

$$|\theta-1|\sim\frac{(U+3/2)M^2}{Z\alpha\eta|M^2-M_*^2|S}\left|\frac{dS}{d\zeta}\right|\ll 1. \qquad (5.2)$$

The condition for the usual equations of gasdynamics to be applied in the description of plasma flows also has the form (2.1) in this case.

Setting the expression for $\theta - 1$ from (5.1) into Eqs. (2.2)–(2.5), (2.8), and (2.9), we obtain the system for a plasma in thermal equilibrium:

$$(M^2-M_*^2)\frac{S}{v}\frac{dv}{dS}=M_*^2,$$

$$(M^2-M_*^2)\frac{S}{\alpha}\frac{d\alpha}{dS}=-\frac{1+\alpha}{\alpha\eta}M^2,$$

$$(M^2-M_*^2)\frac{S}{t}\frac{dt}{dS}=-(M^2-M_*^2)\frac{S}{U}\frac{dU}{dS}= \qquad (5.3)$$

$$=-\frac{M^2}{\eta}\left[1+\frac{2}{U\alpha(1-\alpha)}+\frac{5}{2U}\right],$$

$$(M^2-M_*^2)\frac{S}{M^2}\frac{d(M^2)}{dS}=\left[2+\frac{2}{U\alpha(1-\alpha)}+\frac{5}{2U}\right]\frac{M^2}{\eta}+2M_*^2.$$

Thus it follows that the value $M^2 = M_*^2$ is the critical value for the flow of a plasma in thermal equilibrium. As in the case of a plasma which is not in thermal equilibrium, it lies between the isothermal and adiabatic values. Therefore we obtain the expression for the velocity of sound

$$a^2=\left(1-\frac{2}{5}\frac{U}{\eta}\right)\frac{5}{3}\frac{P}{\rho}=$$

$$=\frac{5+\alpha(1-\alpha)(U+5/2)^2}{3+\alpha(1-\alpha)[(U+3/2)^2+3/2]}\frac{P}{\rho}, \qquad (5.4)$$

(this expression was obtained previously in [3] by another method).

If $\alpha \ll U^{-2}$, then the velocity of sound in a plasma which is in thermal equilibrium coincides with the velocity of sound in a nonionized gas. If $\alpha U^2 \gtrsim 1$ and $U \gg 1$, then (5.4) passes into Newton's formula for isothermal sound. The explanation of this is similar to that given for a plasma which is not in thermal equilibrium. In the case under consideration the critical cross section is situated in the throat of the channel. In the subsonic stream

$$\frac{d(M^2)}{dS} < 0, \quad \frac{dv}{dS} < 0, \quad \frac{dt}{dS} > 0, \quad \frac{d\alpha}{dS} > 0.$$

In the supersonic stream the derivatives reverse sign.

We obtain the following expression for the polarization field strength in a plasma in thermal equilibrium:

$$E_x = -\frac{kt}{eL}\frac{M^2}{(M^2 - M_*^2)\eta}\Big[U + \frac{1}{\alpha} + \frac{5}{2}\frac{2-\alpha}{U\alpha(1-\alpha)} + \frac{25}{4U}\Big]\frac{1}{S}\frac{dS}{d\zeta}. \tag{5.5}$$

In this case the relations between the parameters (2.11) have the form

$$\frac{1+\alpha}{U}(M^2+3) + \frac{6}{5}\alpha = \text{const},$$

$$\frac{\alpha^2}{1-\alpha} = \frac{vS}{U^2 \exp U}[(1+\alpha)M^2]^{1/2}. \tag{5.6}$$

We find yet another integral from the second and third equations of (5.3):

$$U(1+\alpha) + \frac{5}{2}\alpha + 2\ln\frac{\alpha}{1-\alpha} = \text{const}. \tag{5.7}$$

Equations (5.6) and (5.7) allow us to find M^2, U, and α as functions of S. Setting $U(\alpha)$ from (5.7) in (5.4), we find the velocity of sound

$$a^2 = \frac{P}{\rho}\left[5(1+\alpha)^2 + \alpha(1-\alpha)\{(1+\alpha_1)(U_1 + {}^5/_2) + 2L(\alpha)\}^2\right]\times$$
$$\times\left[3(1+\alpha)^2 + \alpha(1-\alpha)[\{(1+\alpha_1)(U_1 + {}^3/_2) + \alpha_1 - \alpha + 2L(\alpha)\}^2 + {}^3/_2\right]^{-1}.$$

$$L(\alpha) = \ln[\alpha_1(1-\alpha)/\alpha(1-\alpha_1)]. \tag{5.8}$$

For steady-state flow in a cylindrical channel we have

$$\left(\frac{1+\alpha}{UM^2}\right)^{1/2}\left(M^2 + \frac{3}{5}\right) = \text{const}. \tag{5.9}$$

Thus we find from the first equation of (5.6)

$$A \equiv \frac{M^2(M^2 + 3 + {}^6/_5U)}{(M^2 + {}^3/_5)^2} = \text{const},$$

$$\left(\max A \ \text{ for } M^2 = \frac{3}{5}\frac{U + {}^5/_2}{U + {}^3/_2}\right). \tag{5.10}$$

This value of M^2 corresponds to the propagation velocity for weak disturbances, so the velocity of sound is

$$a^2 = \frac{U + {}^5/_2}{U + {}^3/_2}\frac{P}{\rho}. \tag{5.11}$$

It follows from the second equation of (5.6) and (5.7) that for $\alpha \ll 1$ the plasma will be barotropic ($P \sim \rho^{5/3}$); in this case we have from (5.6) and (5.7)

$$\alpha^2 \exp U = \text{const}, \quad \frac{U^3}{(M^2)^{1/2}S} = \text{const},$$

$$\frac{M^2+3}{U} + \frac{6}{5}\alpha = \text{const}. \tag{5.12}$$

Thus we have M^2 as a function of S/S*:

$$\frac{(M^2+3)^2}{16(M^2)^{1/2}} = \tag{5.13}$$
$$= \left\{1 + \frac{3}{10}\alpha^* U^*\left[1 - \exp\left\{\frac{U^*}{2}\left[1 - (M^2)^{1/4}\left(\frac{S}{S^*}\right)^{1/2}\right]\right\}\right]\right\}^2 \frac{S}{S^*}.$$

It follows from this formula that when ionization is present the curve S/S* (M^2) lies lower that the corresponding curve for an ordinary gas (when ionization is present an identical increase of Mach number is attained for a smaller variation of channel cross section than for an ordinary gas).

Moreover, we find S/S* as a function of U, and also of α:

$$\frac{S}{S^*} = \left(\frac{U}{U^*}\right)^2\left[4\frac{U}{U^*} - 3 + \frac{6}{5}\alpha^* U\left(1 - \exp\frac{U^* - U}{2}\right)\right]^{-1/2}, \tag{5.14}$$

$$\frac{S}{S^*} = \left(1 + \frac{2}{U^*}\ln\frac{\alpha^*}{\alpha}\right)^2\left\{4\left[1 + \frac{3}{10}(\alpha^* - \alpha)U^*\right]\left(1 + \frac{2}{U^*}\ln\frac{\alpha^*}{\alpha}\right) - 3\right\}^{-1/2}. \tag{5.15}$$

The curve (5.14) also lies lower than the corresponding curve for an ordinary gas (when ionization is present an identical variation of temperature is attained for a smaller variation of channel area than in the case of an ordinary gas).

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